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Riemann scalar curvature of ideal quantum gases obeying Gentile's statistics

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Abstract. The scalar curvature (R) of ideal quantum gases obeying Gentile's statistics is investigated by the method of information geometrical theory. The R value is specified by the fugacity η and the maximum number, p, of particles in a state. The lowest case $p = 1$, corresponds to Fermi–Dirac statistics and the unbounded case, $p \to \infty$, to Bose–Einstein statistics. In contrast to $R = 0$ for ideal classical gases obeying Boltzmann statistics, we find $R = \sqrt{2}/32$ for $p \geqslant 2$ and $R = -\sqrt{2}/32$ for $p = 1$, in $\eta \to 0$ which is the classical limit. This means that a quantum statistical character is left in R, in the classical limit. Also, a correlation between the sign of R and a quantum mechanical exchange effect is recognized for $\eta \to 0$ and $\eta \gg 1$. Furthermore, we obtain results that support the instability interpretation of R proposed by Janyszek and Mrugala.

1. Introduction

Geometrical approaches to thermodynamics have been tried by many authors. Contact geometry was introduced into equilibrium thermodynamics by Hermann [1] and developed by Mrugala [2], Janyszek and Mrugala [3]. Another geometrical approach to equilibrium thermodynamics was based on the concept of the distance between thermodynamic states. Weinhold [4] defined a metric tensor by the second derivatives of the internal energy U, of equilibrium systems with respect to extensive parameters.

Ruppeiner [5] included a fluctuation theory in the axioms of thermodynamics and defined a Riemannian metric on a manifold of equilibrium states. His metric tensor was represented by the second moments of fluctuation and he showed, calculating thermodynamic curvatures (R) of many models, that the R values are equal to correlation volumes near critical points. Later, the two metrics defined by Weinhold and Ruppeiner were shown, by Mrugala [6], Janyszek and Mrugala [7] and Salamon *et al* [8], to be equivalent within a contact transformation.

A statistical approach to the geometry of thermodynamics was initiated by Ingarden [9] and Ingarden *et al* [10]. They defined a metric, making use of a relative entropy. This approach was applied by Janyszek and Mrugala (JM) to real and ideal gases [11], quantum magnetic systems [12] and ideal quantum gases [13]. Studying R of ideal quantum gases obeying the Bose–Einstein statistics (BE) and the Fermi–Dirac statistics (FD), they proposed the instability interpretation of R and also made the hypothesis that the sign of R manifests a quantum mechanical exchange effect.

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On the other hand, a differential information geometrical theory was constructed by Amari [14] and others. In information geometrical theory a metric tensor is defined by the expectation values of the second derivatives of a probability density function with respect to its parameters. The theory contains a generalized connection with a parameter α which is called an α-connection. When $\alpha = 0$, the generalized connection reduces to the so-called Riemann–Christoffel connection.

Recently, geometrical approaches to non-equilibrium systems were studied by Obata *et al* [15] and Obata and Hara [16]. They introduced, by the method of information geometrical theory, a metric on a space of probabilities that characterize random or correlated walks. Through the calculation of R, they concluded that R is a measure of instability in non-equilibrium systems as well as in thermodynamic systems.

In this paper we calculate R, using the method of information geometrical theory, for ideal quantum gases which obey Gentile's statistics [17], which is one of the intermediate or interpolative statistics. In Gentile's statistics, the maximum number of particles in a state is an arbitrary integer p, in contrast with one for FD and infinity for BE. It has recently been shown that ideal q-fermion gases have the same property provided q is complex and has an absolute value of one [18]. The R of these gases is specified by the maximum occupation number p, and the fugacity η. The present calculation is a generalization of JM's work [13] in which η was restricted to $0 < \eta < 1$ and the statistics restricted to BE and FD. Through numerical and analytical calculations of R in the same units as JM's work we get the following results:

(1) In the limit $\eta \to 0$, which is a classical limit, $R \to \sqrt{2}/32$ for $p \geqslant 2$ and $R \to -\sqrt{2}/32$ for $p = 1$ in contrast with $R = 0$ for ideal classical gases [5, 10, 11]. These results show that a quantum statistical characteristic is preserved in R, in the classical limit.
(2) In the intermediate range $0 < \eta \leqslant 1$, R is a monotonically increasing function of p. So we can say that Gentile's statistics is intermediate between BE and FD with respect to R.
(3) R values in $\eta \to 0$ and $\eta \gg 1$ show strong correlations due to a quantum mechanical exchange effect.
(4) R is infinity at $\eta = 1$ for $p \to \infty$ which corresponds to the Bose–Einstein condensation. This supports the instability interpretation of R proposed by JM.

This paper is organized as follows. In section 2 we give a short review of information geometrical theory for later convenience, and in section 3 apply the theory to ideal quantum gases to reproduce the results of JM's work. In section 4 we describe Gentile's statistics and derive a general formula of its R. In section 5 we represent a calculation method for R and give some results. Finally, in section 6 we discuss theses results. Throughout this paper we follow Misner *et al* 's [19] convention for geometrical signatures.

2. Information geometrical theory and thermodynamics

In this section we briefly review the information geometrical theory [14] that is used to geometrically analyse a family of probability density functions (PDF) and its application to thermodynamics. Let $p(x, \theta)$ be a PDF described by a probability variable, x, and parameters $\theta \equiv \{\theta^1, \theta^2, \ldots, \theta^n\}$ that characterize a system. A set of PDFs

$$S \equiv \{p(x, \theta) | \theta \in \Omega\} \qquad \theta \equiv \{\theta^1, \theta^2, \ldots, \theta^n\} \tag{1}$$

becomes an n-dimensional statistical manifold having θ^i coordinates. Ω is a subset in $\mathbb{R}^n$. According to information geometrical theory, we can make a metric tensor $g_{ik}(\theta)$:

$$g_{ik}(\theta) \equiv E[\partial_i l(x, \theta) \partial_k l(x, \theta)] = -E[\partial_i \partial_k l(x, \theta)] \tag{2}$$

where $l(x,\theta) \equiv \ln p(x,\theta)$ and $E[\ ]$ means the expectation operation with respect to $p(x,\theta)$. The last expression is obtained by the use of the normalization condition $E[\partial_i l(x,\theta)] = 0$. This metric tensor is a Fisher information matrix in information theory.

We now restrict our attention to a special family of PDFs, called an exponential family, that is described by

$$p(x,\theta) = \exp\left[C(x) + \sum_{i=1}^{n} \theta^i F_i(x) - \Psi(\theta)\right] \tag{3}$$

where $C(x)$ and $F_i(x)$ are arbitrary functions of x, and $\Psi(\theta)$ is a function of θ^i coordinates. For example, a normal distribution function belongs to the family. The metric tensor of the exponential family is straightforwardly obtained from equation (2) as

$$g_{ik}(\theta) = \frac{\partial^2 \Psi(\theta)}{\partial\theta^i \partial\theta^k} \tag{4}$$

and then the Christoffel connection coefficients are given by

$$\Gamma_{ijk}(\theta) \equiv \frac{1}{2}(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}) = \frac{1}{2}\frac{\partial^3 \Psi(\theta)}{\partial\theta^i \partial\theta^j \partial\theta^k}. \tag{5}$$

In case the parameter space is two-dimensional, the R is reduced to

$$R = -\frac{1}{2\det(g)^2}\begin{vmatrix} g_{11} & g_{12} & g_{22} \\ g_{11,1} & g_{12,1} & g_{22,1} \\ g_{11,2} & g_{12,2} & g_{22,2} \end{vmatrix} = -\frac{1}{2\det(g)^2}\begin{vmatrix} \Psi_{,11} & \Psi_{,12} & \Psi_{,22} \\ \Psi_{,111} & \Psi_{,112} & \Psi_{,122} \\ \Psi_{,112} & \Psi_{,122} & \Psi_{,222} \end{vmatrix} \tag{6}$$

where $(,i)$ means the derivative with respect to θ^i, $i = 1, 2$. Note that, in this case R is constituted by up to the first derivative of the metric tensor or the third derivatives of $\Psi(\theta)$ because of the symmetry of the metric tensor and the connection coefficients.

3. Scalar curvature of ideal Bose gases and Fermi gases

The PDF of a grand canonical ensemble is represented by

$$p(x,\theta) = \frac{\exp[\alpha N - \beta E]}{\Xi} = \exp[\alpha N - \beta E - \ln \Xi] \tag{7}$$

where $\beta \equiv 1/kT$, k is the Boltzmann constant, T the temperature, $\alpha \equiv \mu/kT$, μ the chemical potential and Ξ the grand partition function. Comparing equation (7) with (3), we see that the family of PDFs is a two-dimensional exponential family with coordinates α, β. Thus its metric tensor can be calculated with formula (4), that is,

$$g_{ik}(\theta) = \frac{\partial^2 \ln \Xi}{\partial\theta^i \partial\theta^k} \qquad (\theta^i = \alpha, \beta). \tag{8}$$

The grand partition functions Ξ for ideal gases obeying BE and FD are given by

$$\ln \Xi = Vg\frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\frac{2}{3}\beta\int_0^\infty \frac{1}{\mathrm{e}^{\beta\varepsilon-\alpha} \pm 1}\varepsilon^{3/2}\,\mathrm{d}\varepsilon \qquad (+ \text{ for FD and } - \text{ for BE}) \tag{9}$$

where V is the volume, m the mass of a particle and $g \equiv 2s + 1$ where s is its spin. The rhs can be rewritten as

$$\ln \Xi = g\lambda^{-3} V h_{5/2}(\eta) \tag{10}$$

using the mean thermal wavelength $\lambda \equiv h/\sqrt{2\pi mkT}$, the fugacity $\eta \equiv \mathrm{e}^\alpha = \mathrm{e}^{\mu/kT}$ and $h_l(\eta)$ that are defined by

$$h_l(\eta) \equiv \frac{1}{\Gamma(l)}\int_0^\infty \frac{x^{l-1}}{\eta^{-1}\mathrm{e}^x - 1}\,\mathrm{d}x. \tag{11}$$

Table 1. R values for BE and FD in units of $5\lambda^3/Vg$. Corrections are made for numerical errors for FD displayed on JM's table 1 [13].

η	BE	FD
0.100	0.4539×10^{-1}	-0.4316×10^{-1}
0.300	0.4852×10^{-1}	-0.4146×10^{-1}
0.500	0.5337×10^{-1}	-0.4010×10^{-1}
0.700	0.6245×10^{-1}	-0.3900×10^{-1}
0.900	0.9187×10^{-1}	-0.3807×10^{-1}
0.910	0.9563×10^{-1}	-0.3802×10^{-1}
0.920	0.1001	-0.3798×10^{-1}
0.930	0.1054	-0.3794×10^{-1}
0.940	0.1121	-0.3790×10^{-1}
0.950	0.1207	-0.3786×10^{-1}
0.960	0.1323	-0.3781×10^{-1}
0.970	0.1493	-0.3777×10^{-1}
0.980	0.1778	-0.3773×10^{-1}
0.990	0.2423	-0.3769×10^{-1}

$\Gamma(l)$ is the gamma function and $h_l(\eta)$ have the following recursion relations:

$$h_{l-1}(\eta) = \eta \frac{\mathrm{d}h_l(\eta)}{\mathrm{d}\eta}. \tag{12}$$

The metric tensor and its first derivatives are functions of $h_{5/2}(\eta)$, $h_{3/2}(\eta)$, $h_{1/2}(\eta)$, $h_{-1/2}(\eta)$ and R can be calculated by using equation (6):

$$R = 5\frac{\lambda^3}{Vg} \frac{h_{3/2}^2(\eta)h_{1/2}(\eta) - 2h_{5/2}(\eta)h_{1/2}^2(\eta) + h_{5/2}(\eta)h_{3/2}(\eta)h_{-1/2}(\eta)}{[5h_{5/2}(\eta)h_{1/2}(\eta) - 3h_{3/2}^2(\eta)]^2}. \tag{13}$$

JM [13] executed the series expansion of $h_l(\eta)$ with respect to η under the condition $0 < \eta < 1$ and obtained the result

$$h_l(\eta) = \sum_{j=1}^{\infty}(\pm 1)^{j-1}\frac{\eta^j}{j^l} \qquad (- \text{ for FD and + for BE}). \tag{14}$$

By a numerical method, they also obtained R for BE and FD. We have performed the same numerical calculation in order to examine the extraordinarily large values of R for FD displayed in JM's table 1 [13]. We found that their table includes large errors for FD. Our result for R in units of $5\lambda^3/Vg$ is displayed in table 1 where numerical errors for FD are corrected. Table 1 shows that R for BE is positive and monotonically increasing to infinity as η tends to one, whereas R for FD is negative and very slowly increasing.

As fluctuations in single-phase systems are thermodynamically negligible, such systems are relatively stable. Fluctuations become very important in multiphase systems, especially in the vicinity of the critical points. As a result, in the closest vicinity of the critical points, systems become extremely unstable. Since R depends on the second and third moments of fluctuations, JM interpreted R as a measure of global fluctuations in the systems caused by quasi-interactions. From this, they insisted that R is a measure of the instability of systems. Using this interpretation, they explained that the divergence of R at $\eta = 1$ for BE indicates the unstableness of the Bose gases in the region where Bose–Einstein condensation occurs. On the other hand, they insisted that the smallness of R in the classical limit $\eta \to 0$ corresponds to the stableness of Bose gases because the systems are far from the region in which Bose–Einstein condensation occurs. Furthermore, they noted the sign of R: $R > 0$ for BE manifests the quantum mechanical exchange effect as attractive, and with $R < 0$ for FD, repulsive.

4. Scalar curvature of Gentile's statistics

JM calculated R of ideal quantum gases and suggested the instability interpretation of R. However, the study was restricted to $0 < \eta < 1$ and only to BE and FD. The R of ideal quantum gases obeying intermediate statistics and R in $1 < \eta$ have been left unsolved. To generalize JM's work, we choose Gentile's statistics [17], which is the simplest of all the intermediate statistics. The grand partition function Ξ of ideal quantum gases obeying Gentile's statistics is given by

$$\ln \Xi = Vg \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \frac{2}{3}\beta \int_0^\infty \left[\frac{1}{e^{\beta\varepsilon-\alpha} - 1} - \frac{p+1}{e^{(p+1)(\beta\varepsilon-\alpha)} - 1} \right] \varepsilon^{3/2}\, d\varepsilon \qquad (15)$$

where p represents the maximum number of particles that can occupy a single state. As is easily confirmed, the function for $p = 1$ reduces to the grand partition function for FD and $p \to \infty$ reduces to that for BE. Following equation (10), we represent the grand partition function as

$$\ln \Xi = Vg\lambda^{-3} G_{5/2}(\eta) \qquad (16)$$

where the function $G_l(\eta)$ is defined by

$$G_l(\eta) \equiv \frac{1}{\Gamma(l)} \int_0^\infty \left[\frac{1}{\eta^{-1}e^x - 1} - \frac{p+1}{\eta^{-(p+1)}e^{(p+1)x} - 1} \right] x^{l-1}\, dx. \qquad (17)$$

This is a generalization of equation (11). The R of ideal quantum gases obeying Gentile's statistics is given by replacing $h_l(\eta)$ with $G_l(\eta)$ in equation (13):

$$R = 5\frac{\lambda^3}{Vg} \frac{G_{3/2}^2(\eta)G_{1/2}(\eta) - 2G_{5/2}(\eta)G_{1/2}^2(\eta) + G_{5/2}(\eta)G_{3/2}(\eta)G_{-1/2}(\eta)}{[5G_{5/2}(\eta)G_{1/2}(\eta) - 3G_{3/2}^2(\eta)]^2}. \qquad (18)$$

5. Numerical and analytical calculation of R

We now numerically calculate R in the range $0.1 < \eta < 3.0$, and for $\eta \to 0$ and $1 \ll \eta$ by using a series expansion of $G_l(\eta)$. In the numerical calculation we transform the integrand of equation (17) to

$$G_{5/2}(\eta) \equiv \frac{1}{\Gamma(5/2)} \int_0^\infty \frac{\sum_{k=1}^{p} k y^{p-k}}{\sum_{k=0}^{p} y^k} x^{3/2}\, dx \qquad \left(y \equiv \frac{e^x}{\eta} \right) \qquad (19)$$

to avoid an apparent singularity at $e^x = \eta$. By the transformation, the integrand of equation (19) becomes finite at $e^x = \eta$ when $p < \infty$ and infinite when $p \to \infty$. The infinity corresponds to the fact that the integrand of the grand partition function for BE is infinite at $e^x = \eta$. The integration is numerically calculated by the Simpson method. The other components $G_{3/2}(\eta)$, $G_{1/2}(\eta)$, $G_{-1/2}(\eta)$ can be obtained straightforwardly through equation (12).

Figure 1 shows R values in units of $5\lambda^3/Vg$ in the range $0.1 < \eta < 3.0$ for $p = 1, 5, 10, 20, 30$ and also R for $p \to \infty$. The numerical data for $p = 1, \infty$ is listed in table 1. Figure 2 is a three-dimensional plot of R in the ranges $0.1 < \eta < 3.0$ and $1 \leqslant p \leqslant 30$.

In the range $\eta < 1$, R is a monotonically increasing function of p. Thus we can say that Gentile's statistics is intermediate between FD and BE with respect to R. For BE the infinite R at $\eta = 1$ corresponds to the Bose–Einstein condensation, whereas in the case of Gentile's statistics for $p < \infty$, the finiteness of R at $\eta = 1$ suggests the stableness of systems because the occupation number of particles is finite at zero energy and so the Bose–Einstein condensation cannot happen.

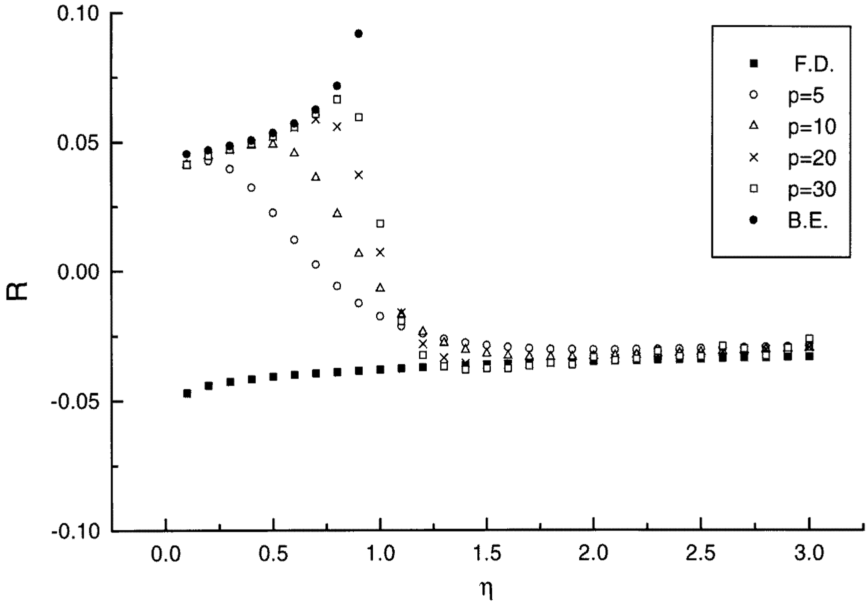

Figure 1. R values for GS in the range $0.1 < \eta < 3.0$ for $p = 1, 5, 10, 20, 30$ and R for $p \to \infty$ in units of $5\lambda^3/Vg$.

The R values for the classical limit $\eta \to 0$ and the high-density limit $1 \ll \eta$ are not explicitly shown in figure 1 because of a lack of numerical precision. Here we show results of analytical calculations in the limits.

First, in spite of the fact that R is zero for ideal classical gases, we show that the R values of BE and FD in the limit $\eta \to 0$ are different from that of ideal classical gases. Expanding $G_l(\eta)$ in series of η under the condition $0 < \eta < 1$, we get

$$G_l(\eta) = \sum_{j=1}^{\infty} \left\{ \eta^j \left(\frac{1}{j}\right)^l - (p+1)\eta^{j(p+1)} \left(\frac{1}{j(p+1)}\right)^l \right\} \tag{20}$$

and approximating the functions up to $O(\eta^2)$, we obtain

$$G_l(\eta) = \eta + \eta^2 \left(\frac{1}{2}\right)^l - (p+1)\eta^{p+1} \left(\frac{1}{p+1}\right)^l + O(\eta^3). \tag{21}$$

Comparing these functions with equation (14), we notice that these functions reduce to $h_l(\eta)$ of FD for $p = 1$ and that of BE for $p \geqslant 2$. In consequence R calculated by equation (18) takes two different values depending on $p = 1$ or $p \geqslant 2$ in the limit $\eta \to 0$. Straightforward calculation leads us to the following two values of R in units of $5\lambda^3/Vg$:

$$\lim_{\eta \to 0} R = \begin{cases} +\dfrac{\sqrt{2}}{32} & (p \geqslant 2) \\ -\dfrac{\sqrt{2}}{32} & (p = 1) \end{cases} \tag{22}$$

that are different from $R = 0$ of ideal classical gases obeying Boltzmann statistics. This is a new result as far as we know. This difference originates from the fact that the lowest order terms of $O(\eta^3)$ in the numerator of equation (18) are completely cancelled out and so the next order terms of $O(\eta^4)$ involving statistical differences become leading terms.

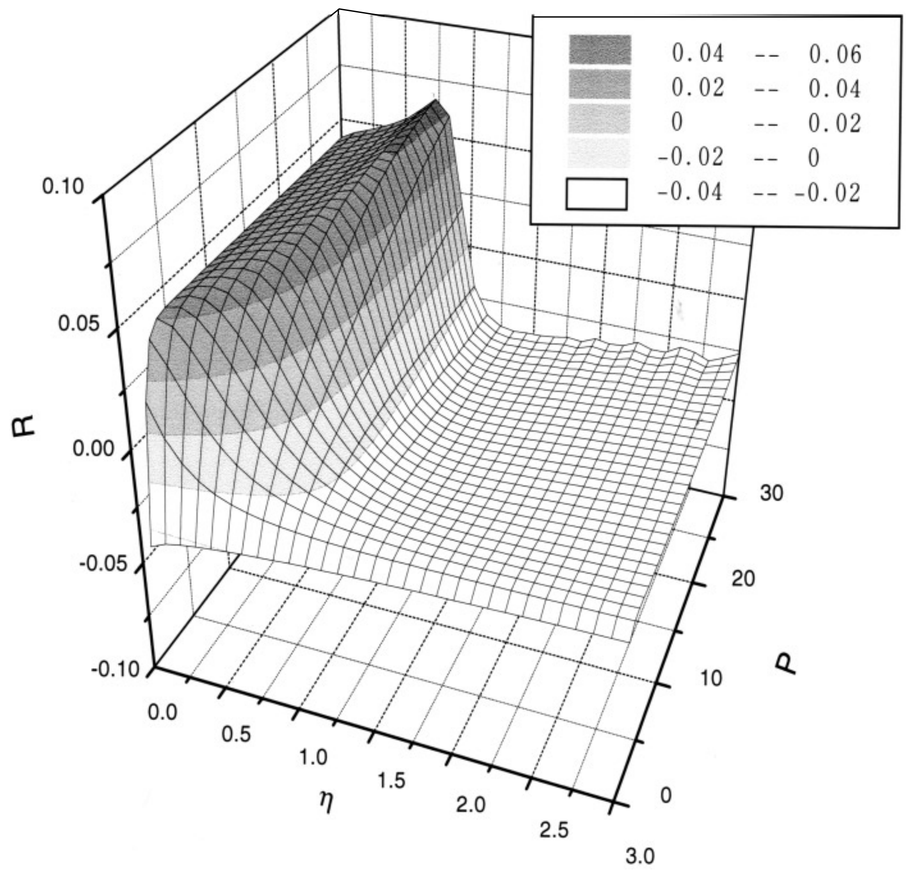

Figure 2. A three-dimensional plot of R for GS in the ranges $0.1 < \eta < 3.0$ and $1 \leqslant p \leqslant 30$.

To explain the matter in detail, we write the grand partition function of ideal classical gases:

$$\ln \Xi = Vg \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \frac{2}{3}\beta \int_0^\infty \frac{1}{e^{\beta\varepsilon-\alpha}} \varepsilon^{3/2} \, d\varepsilon \tag{23}$$

as

$$\ln \Xi = g\lambda^{-3} V B_{5/2}(\eta) \qquad B_l(\eta) \equiv \frac{1}{\Gamma(l)} \int_0^\infty \frac{1}{\eta^{-1}e^x} x^{l-1} \, dx. \tag{24}$$

Partial integration leads to

$$B_{5/2}(\eta) = B_{3/2}(\eta) = B_{1/2}(\eta) = B_{-1/2}(\eta) = \eta. \tag{25}$$

These functions are just the first term in the rhs of equation (21), and by replacing $h_l(\eta)$ by $B_l(\eta)$ in equation (13) we get $R = 0$ independently of η. For this reason the R of ideal quantum gases obeying Gentile's statistics is not equal to that of ideal classical gases.

Next we reconsider, from the thermodynamic point of view, the reason why the R values of Gentile's statistics in the limit $\eta \to 0$ are divided into two groups, depending on $p = 1$ or $p \geqslant 2$. Taking advantage of the well known relations between the grand partition function and thermodynamic quantities, we have

$$PV = kT \ln \Xi = kTVg\lambda^{-3}G_{5/2}(\eta)$$
$$= kTVg\lambda^{-3}\left(\eta + \eta^2\left(\frac{1}{2}\right)^{5/2} - (p+1)\eta^{p+1}\left(\frac{1}{p+1}\right)^{5/2} + \mathrm{O}(\eta^3)\right) \qquad (26a)$$

$$N = kT\left(\frac{\partial}{\partial\mu}\ln\Xi\right)_{T,V} = Vg\lambda^{-3}G_{3/2}(\eta)$$
$$= Vg\lambda^{-3}\left(\eta + \eta^2\left(\frac{1}{2}\right)^{3/2} - (p+1)\eta^{p+1}\left(\frac{1}{p+1}\right)^{3/2} + \mathrm{O}(\eta^3)\right) \qquad (26b)$$

where P is the pressure and N the number of particles. From (26a) and (26b) we get up to $\mathrm{O}(\eta)$

$$\frac{PV}{kTN} = \begin{cases} 1 - \left(\frac{1}{2}\right)^{5/2}\eta & (p \geqslant 2) \\ 1 + \left(\frac{1}{2}\right)^{5/2}\eta & (p = 1) \end{cases}$$

and using N instead of η, we obtain

$$\frac{PV}{kTN} = \begin{cases} 1 - \frac{N}{Vg}\frac{\hbar^3}{2}\left(\frac{\pi}{mkT}\right)^{3/2} & (p \geqslant 2) \\ 1 + \frac{N}{Vg}\frac{\hbar^3}{2}\left(\frac{\pi}{mkT}\right)^{3/2} & (p = 1). \end{cases} \qquad (27)$$

These equations represent the state equations of ideal quantum gases obeying Gentile's statistics in the limit $\eta \to 0$, and the second terms are regarded as corrections for pressure or quantum mechanical exchange effects [20]. So we can say that the quantum mechanical exchange effect is repulsive for $p = 1$ and attractive for $p \geqslant 2$ as well as BE. On the other hand, as stated above, R in the limit $\eta \to 0$ is negative for $p = 1$ and positive for $p \geqslant 2$. These facts support the hypothesis made by JM: the difference of the sign of R is caused by the quantum mechanical exchange effect.

Secondly we examine R in $1 \ll \eta$. As seen from figure 1, R for ideal quantum gases, except BE, takes a negative value when $1 < \eta$, and seems to approach zero from the negative side when $1 \ll \eta$. We analytically show such characteristics of R.

Expanding $G_{5/2}(\eta)$ in series with respect to α for $\alpha \gg 1$, we get

$$G_{5/2}(\alpha) = \frac{1}{\Gamma(5/2)}\left\{\frac{2}{5}p\alpha^{5/2} + \frac{\pi^2}{2}\frac{p}{p+1}\alpha^{1/2} - \frac{\pi^4}{120}\left[1 - \frac{1}{(p+1)^3}\right]\alpha^{-3/2} + \mathrm{O}(\alpha^{-7/2})\right\}. \qquad (28)$$

Differentiating $G_{5/2}(\alpha)$ with respect to α, we can get the other functions $G_{3/2}(\alpha)$, $G_{1/2}(\alpha)$, $G_{-1/2}(\alpha)$:

$$G_{3/2}(\alpha) = \frac{1}{\Gamma(5/2)}\left\{p\alpha^{3/2} + \frac{\pi^2}{4}\frac{p}{p+1}\alpha^{-1/2} + \frac{\pi^4}{80}\left[1 - \frac{1}{(p+1)^3}\right]\alpha^{-5/2} + \mathrm{O}(\alpha^{-9/2})\right\}$$
$$G_{1/2}(\alpha) = \frac{1}{\Gamma(5/2)}\left\{\frac{3}{2}p\alpha^{1/2} - \frac{\pi^2}{8}\frac{p}{p+1}\alpha^{-3/2} + \mathrm{O}(\alpha^{-7/2})\right\} \qquad (29)$$
$$G_{-1/2}(\alpha) = \frac{1}{\Gamma(5/2)}\left\{\frac{3}{4}p\alpha^{-1/2} + \frac{3\pi^2}{16}\frac{p}{p+1}\alpha^{-5/2} + \mathrm{O}(\alpha^{-9/2})\right\}.$$

Using these functions, we see that R in $1 \ll \alpha$ is as follows:

$$R = -\frac{1}{5}\Gamma\left(\frac{5}{2}\right)\left[\pi^2 \frac{p}{p+1}\sqrt{\alpha}\right]^{-1}. \tag{30}$$

Then R for $p < \infty$ approaches zero from the negative side in $1 \ll \alpha$ or $1 \ll \eta$. In this limit, in turn, Gentile's statistics, except $p \to \infty$, behaves like FD with respect to R.

We try to explain its reason by extending the derivation of thermodynamic formulae in the classical limit $\eta \to 0$ to the other limit $1 \ll \alpha$. In that limit we get

$$\begin{aligned} PV &= kT \ln \Xi = kTVg\lambda^{-3} G_{5/2}(\alpha) \\ &= kTVg\lambda^{-3}\frac{1}{\Gamma(5/2)}\left\{\frac{2}{5}p\alpha^{5/2} + \frac{\pi^2}{2}\frac{p}{p+1}\alpha^{1/2} - \frac{\pi^4}{120}\left[1 - \frac{1}{(p+1)^3}\right]\alpha^{-3/2}\right\} \end{aligned} \tag{31}$$

$$\begin{aligned} N &= kT\left(\frac{\partial}{\partial \mu}\ln \Xi\right)_{T,V} = Vg\lambda^{-3}G_{3/2}(\alpha) \\ &= Vg\lambda^{-3}\frac{1}{\Gamma(5/2)}\left\{p\alpha^{3/2} + \frac{\pi^2}{4}\frac{p}{p+1}\alpha^{-1/2} + \frac{\pi^4}{80}\left[1 - \frac{1}{(p+1)^3}\right]\alpha^{-5/2}\right\} \end{aligned} \tag{32}$$

or

$$\frac{PV}{kTN} = \frac{2}{5}\alpha\left(1 + \frac{\pi^2}{p+1}\alpha^{-2}\right). \tag{33}$$

Elimination of α from equation (33) with the aid of equation (32) leads to the state equation

$$P = \frac{1}{5}\left(\frac{6\pi^2}{pg}\right)^{2/3}\frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{5/3}\left\{1 + \frac{5}{6}\frac{\pi^2}{p+1}\alpha_0^{-2}\right\} \tag{34}$$

where

$$\alpha_0 = \left[\frac{N\lambda^3}{pVg}\Gamma(\tfrac{5}{2})\right]^{2/3} \tag{35}$$

is the first-order solution of equation (32). The main term of equation (34) for $p = 1$ is written as

$$P = \frac{1}{5}\left(\frac{6\pi^2}{g}\right)^{2/3}\frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{5/3} \qquad (p = 1)$$

which is just the state equation of degenerated Fermi gases [20], so we can regard the second term in the rhs of equation (34) as a correction for pressure at $1 \ll \alpha$, and the quantum mechanical exchange effect should be noted as repulsive. Hence, this case also supports the hypothesis proposed by JM: $R < 0$ corresponds to the quantum mechanical repulsive forces.

6. Conclusion

A family of grand partition functions for ideal quantum gases becomes a two-dimensional space having coordinates $\alpha \equiv \mu/kT$ and $\beta \equiv 1/kT$. This family is an exponential family in the geometrical theory of information, so we were able to define a metric tensor and calculate R.

JM studied R for FD and BE and concluded that a bigger R means a less stable state of systems, and also made the hypothesis that the sign of R in the $\eta \to 0$ limit manifests a quantum mechanical exchange effect. This conclusion and hypothesis are attractive. However, possibilities have been left that a diverging R occurs in other cases and that the change of the sign of R would be an accident. Motivated by this, we studied whether this conclusion and hypothesis are available to an intermediate statistics and also investigated the case $\eta > 1$ which JM never referred to. Specifically, we adopted Gentile's statistics, which is the simplest of all intermediate statistics. In Gentile's statistics, the maximum number of particles in a state is an arbitrary integer p, in contrast with one for FD and infinity for BE, and R is specified by p and η in units of $5\lambda^3/Vg$. We studied R numerically for $0.1 < \eta < 3.0$ and analytically for $\eta \to 0$ and $1 \ll \eta$ by expanding, in series, the partition function of Gentile's statistics, and obtained the following results.

In the range $0.1 < \eta < 3.0$, R values for $p \geqslant 2$ converge to that of BE ($p \to \infty$) when η takes small values, and gradually rise as η is increased, but start to decrease at around $\eta \approx 1.0$, above which the R values become negative and finally converge to that of FD ($p = 1$) in the limit $1 \ll \eta$. In this range, R monotonically changes with p and takes values between FD and BE, so we can say that Gentile's statistics is intermediate between BE and FD with respect to R.

In the limit $\eta \to 0$, R for $p = 1$ and $p \geqslant 2$ take different values such as

$$\lim_{\eta\to 0} R = \begin{cases} +\dfrac{\sqrt{2}}{32} & (p \geqslant 2) \\ -\dfrac{\sqrt{2}}{32} & (p = 1). \end{cases}$$

These values are not consistent with $R = 0$ for ideal classical gases. Hence we can say that a quantum statistical property is maintained in R, even in the classical limit. We also investigated a quantum mechanical exchange effect through the state equations in the classical limit, and found that the effect is repulsive for $p = 1$ and attractive for $p \geqslant 2$ as well as $p \to \infty$. The fact that the two values of R exist in the limit $\eta \to 0$ is consistent with the quantum mechanical exchange effect in the limit, that is repulsive when $R < 0$ and attractive when $R > 0$.

Such a quantum mechanical exchange effect was also examined in the limit $1 \ll \eta$, and we found that the effect is repulsive for all p except $p \to \infty$. Again the result is in conformity with the fact that $R < 0$ for all p except $p \to \infty$ in the limit $1 \ll \eta$.

Furthermore, we never observed any divergence of R except for the infinity at $\eta = 1$ for $p \to \infty$. In other words, the divergence of R occurs only for BE. From this observation, we think that the divergence of R for BE corresponds to Bose–Einstein condensation at the critical point and manifests the unstableness of systems, as suggested by JM.

From these results, we conclude that the sign of R for ideal quantum gases obeying Gentile's statistics as well as FD and BE manifests a quantum mechanical exchange effect and that larger values of R correspond to unstable systems.

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